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Explicit, rigorous solutions to two-dimensional heat transfer: two-component media and optimization of cooling fins

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Abstract—New analytical solutions to the problem of steady heat conduction from the wall with longitudinal fins to the environment are derived. Within the two media two temperature fields are harmonic functions with rigorous conjugation of temperature and normal flux along the interface between the two components. First, for high values of the ratio $e = k_1/k_2$, with k_1 and k_2 being thermal conductivities of the grooved wall and environment, respectively, we derive the optimal fin contour providing extreme heat flux (total heat dissipation) from the fin surface at prescribed fin cross-sectional area. This optimizer is found in the class of arbitrary curves and both necessary and sufficient extremum conditions are satisfied. The extreme line coincides with the contour of constant hydraulic gradient calculated by Polubarinova–Kochina for a seepage flow under a concrete dam. At arbitrary e the same isoperimetric problem is solved in the class of elliptic fins assuming fin spacing large enough to consider an isolated profile. Two non-trivial local extrema exist depending on e. For arbitrary e the case of long rectangular fins with arbitrary direction of the outer field is studied. Streamline refraction illustrates non-trivial fluxes near the finger tips and roots.

1. INTRODUCTION

Intuitively engineers understand very well that extended surfaces exhibit better heat dissipation properties than flat walls or pipes. So grooved boundaries of contact between two media with different conductivity values like metal–air are widely used in practice. Duffin [1] seems to be the first who used a rigorous mathematical model to optimize the shape of a cooling fin (rib, protrusion, groove). His idea was developed for various wall geometries and models [2–5]. In the simplest case, optimization is described by an isoperimetric problem: what is the shape of a fin and fin spacing that provides maximum value of the total heat dissipation at prescribed fin volume (cross-sectional area)?

In all of the bibliography available to us, fin optimization was done in terms of one-dimensional models, i.e. temperature did not vary across the fin. Surprisingly, to our knowledge the powerful technique developed for two-dimensional, three-dimensional optimal shape design, in other fields of continuum mechanics [6, 7], not adequately utilized to problems of fin optimization. Meanwhile, Aziz [2] wrote: 'an interesting area that awaits exploration is the optimization of fins incorporating a two-dimensional (2-D) conduction model.'

In this paper we solve 2-D problems of optimization of cooling fins. We conserve the following assumptions of Aziz:

(1) heat conduction is steady-state;
(2) conductivity of the fin and its base is constant;
(3) there is no heat generation in the fin;
(4) there is perfect thermal contact between the base of the fin and the primary surface.

In contrast with the previous works mentioned we do not assume that environment temperature T_2 is constant and that heat flux is proportional to the difference between T_2 and the fin temperature T_1. We assume the Fourier law $v = -k\nabla T$ where v is the flux vector and hence search for two harmonic temperature functions

$$\Delta T_1 = 0, \quad \Delta T_2 = 0 \tag{1}$$

within the two media of constant conductivities k_1 and k_2. Temperatures conjugate along the interface between the base (fin) and environment according to the routine conditions:

$$T_1 = T_2, \quad k_1 \partial T_1/\partial n = k_2 \partial T_2/\partial n. \tag{2}$$

Mathematical equivalence between thermal, electrical and fluid concentration fields in non-homogeneous media described by equations (1) and (2) allows for utilization of interdisciplinary analogies and extension of some known solutions from these specific areas to the heat transfer problems. In what follows we shall employ the schemes and results considered in hydrology to obtain simple analytical solutions to the isoperimetric problems of optimal finning. For this purpose we use the results by Kacimov [8],

NOMENCLATURE

T_1, T_2	fin and environment temperature	x, y	cartesian coordinates
k_1, k_2	fin and environment conductivity	a, b	elliptic fin semiaxes
L	fin spacing	$2h$	width of rectangular fins
Q	total flux through one element of periodic structure	u, v	horizontal and vertical components of thermal gradient.
q	total flux through a fin		
J	heat flux in the environment sufficiently far from fins	Greek symbol	
S	fin cross-sectional area	μ	shape factor.

Kacimov and Obnosov [9], Obdam and Veling [10] and Obnosov [11].

2. IDEALLY CONDUCTIVE WALL

Consider a wall with longitudinal fins spaced distance L apart (Fig. 1). For simplicity we assume that a fin BFC is symmetric about the Oy-axis. Denote the cross-sectional area confined by BFC as S. We prescribe the value of thermal gradient, J, at $y \to \infty$. Total flux passing through one element of our structure is $Q = k_2JL$. In this section we assume that $k_1 \gg k_2$ and hence the boundary ABFCD is isothermic (say $T_2 = 0$ along this line). Therefore we retain only the second equation in (1). Near the fin T_2 is essentially 2-D. We designate as q the total value of flux passing through the fin (this quantity is expressed as an integral of $k_2\nabla T_2$ along BFC). Obviously, $Q-q$ corresponds to the flux from the flat part of the wall.

How much heat can be dissipated by arbitrary fin of prescribed volume? To answer this question we solve the following optimization problem:

Problem 1

For given L, Q, q determine the fin shape providing extreme value of S. It is easy to show that the problem under consideration is equivalent to one solved by Kacimov [8] for a groundwater flow. Namely, Q corresponds to the total flow rate through one strip of the system, and q is equivalent to the asymptotic plume size.

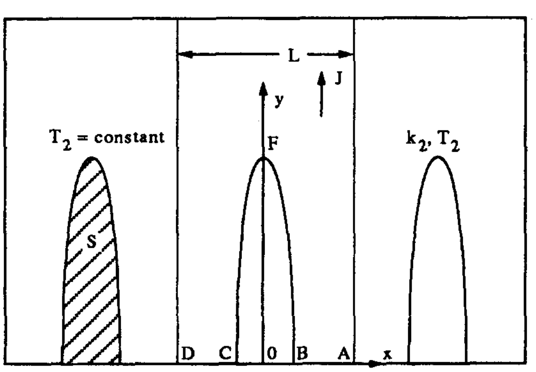

Fig. 1. Periodic system of fins.

Hence the unique, global maximum in problem 1 can be written in explicit form

$$S_{\max} = -\frac{L^2}{4\pi}\ln\cos\frac{\pi q}{2Q}.$$

Note, that both the necessary and sufficient conditions to this maximum are rigorously satisfied.

The left half CF of the extreme contour is described by the parametric equations

$$y = -\frac{L}{4\pi}\ln\frac{\alpha+\cos(\theta)}{\alpha-\cos(\theta)},$$

$$x = -\frac{L}{2\pi}\tan^{-1}\frac{\sin\theta}{\sqrt{\alpha^2-1}}, \quad \pi/2 \leqslant \theta \leqslant \pi \quad (3)$$

where $\alpha = \operatorname{cosec}(\pi q/2Q)$.

This curve coincides with the Polubarinova-Kochina [16] contour of a concrete dam of constant hydraulic gradient. Note, that this contour exhibits other interesting extreme features [8, 17]. At sufficiently large S curve (3) is close to an ellipse which semiaxes are plotted by Polubarinova-Kochina [16].

3. SINGLE FIN WITH ARBITRARY CONDUCTIVITY RATIO

Let us consider the limiting case when L is large as compared with fin sizes such that we can consider a single fin. In contrast with the previous section conductivity ratio is arbitrary. Let the outer thermal gradient, J, sufficiently far from the fin, be perpendicular to the flat interface between the two media. Finally, assume temperature along the line ABOCA (Fig. 2) to be constant (this approximation is often made in 1-D models, see Aziz [2, 5]). Then, we can search for T_1 and T_2 according to equations (1) and (2) in the upper half-plane.

Though this problem seems very simple there are few analytic solutions which involve conjugation conditions in equation (2) in a rigorous form. More precisely, for the 2-D case we know only one solution for a single elliptical (in particular, circular) inclusion [10,

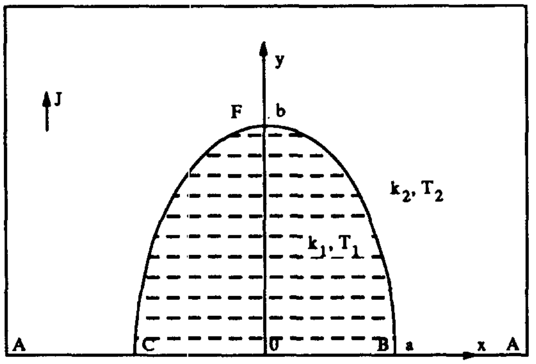

Fig. 2. Elliptical fin with imposed gradient oriented along a semiaxis.

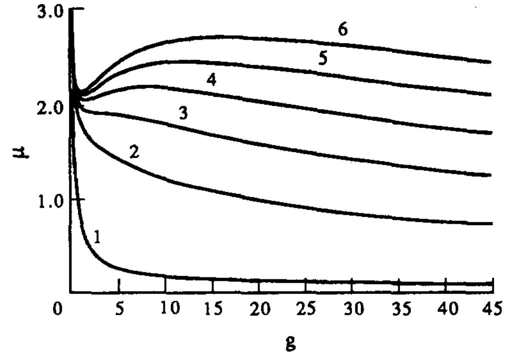

Fig. 3. Shape factor (nondimensional total flux through the fin) as a function of the axis ratio at different values of the conductivity ratio. Curves 1–6 correspond to $e = 0.5$, 4.5, 8.5, 12.5, 16.5, 20.5, respectively.

12] placed within the matrix of different conductivity. Though review of solutions for piece-wise harmonic fields is not within the scope of our paper, we would like to mention ones for two circles and a checkerboard structure [13–15]. In what follows we shall just optimize an elliptical fin for a special case when the outer field is oriented along one of the two axes of the ellipse with semiaxes a and b. Within the fin T_1 varies only with y (isotherms are shown in Fig. 2 by dashed lines) while T_2 is two-dimensional.

The known one-dimensional model [4, 5] is based on the governing equation and boundary conditions:

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[x(y)\frac{\mathrm{d}T}{\mathrm{d}y}\right] = \beta T, \quad T_{\mathrm{BOC}} = \text{const}, \quad (\mathrm{d}T/\mathrm{d}y)_F = 0$$

where $x(y)$ is the fin contour and β is a constant depending on the third type boundary condition along BFC. Noteworthy, that substituting of the rigorous 2-D, two-component solution for a semi-elliptical fin into this equation (recall the condition $T_1 = T_2 = \text{const}$ along ACOBA!) will result in discrepancy βT.

To estimate temperature conduction from the fin we solve the following problem:

Problem 2

At the given k_1, k_2, J and the fin area S, define the axis ratio $g = b/a$ that gives an extreme value of total flux q through the fin.

In the previous section we ignored temperature distribution within the fin and optimized its shape in the class of arbitrary profiles. Here we search for the optimal semi-elliptical fin.

According to Obdam and Veling [10], q is

$$q = ek_2\frac{a(a+b)}{ea+b},$$

then, the shape factor μ is

$$\mu = \frac{q}{k_2\sqrt{S}} = \frac{k_2 e(g+1)}{\sqrt{\pi g}(e+g)}. \tag{4}$$

Obviously, only $g \geqslant 0$ and $e \geqslant 0$ are physically meaningful. By differentiating equation (4) we obtain that the equation $\mathrm{d}\mu/\mathrm{d}g = 0$ has two roots $g_{1,2} = [e-3\pm\sqrt{(e-1)(e-9)}]/2$. Hence, for $e < 9$ the function $\mu(g)$ is monotonic and problem 2 has no solutions. For $e > 9$ our problem exhibits two extrema, namely a local minimum and a local maximum. Obviously, if we impose an additional restriction in problem 2, for instance $g \leqslant$ const, the local extremum can become a global one. At $e \to \infty$ we obtain $g_{\max} \to \infty$ and $g_{\min} \to 1$. In other words, the local minimum passes into a global one. It coincides with results from the previous section at $L \to \infty$ when the optimal ideally conductive fin obtained in the class of arbitrary curves is a semi-circle [18].

In Fig. 3 the functions $\mu(g)$ are plotted (curves 1–6 correspond to conductivities ratio $e = 0.5$, 4.5, 8.5, 12.5, 16.5, 20.5, respectively) illustrating appearance and behavior of the extrema (marked points on the graphs). Local maxima in these curves show that for sufficiently conductive fins their shaping can improve dissipation properties, as compared with close forms.

The question is open whether relations and graphs presented may be used as estimations of q for non-elliptical fins.

4. PERIODIC RECTANGULAR FINS

Let us consider a wall grooved by a periodic system of rectangular fins of width $2h$, length p and step L (Fig. 4a). For arbitrary L and h it seems very difficult to find an analytical solution to problem (1) and (2) and we employ the results obtained by Obnosov [11] and developed by Kacimov and Obnosov [9]. Namely, we study a special case when $L = 2h$, $p \to \infty$. Clearly, this geometry is very specific. However, it is compensated by simplicity of explicit solutions that enable to elucidate 'fine' characteristics of temperature distributions in the two media.

We assume that the flux $J(u_\infty, v_\infty)$ is fixed in the environment at $x \to -\infty$. From Obnosov [11] we can write out the horizontal and vertical components of the gradient at $e > 1$

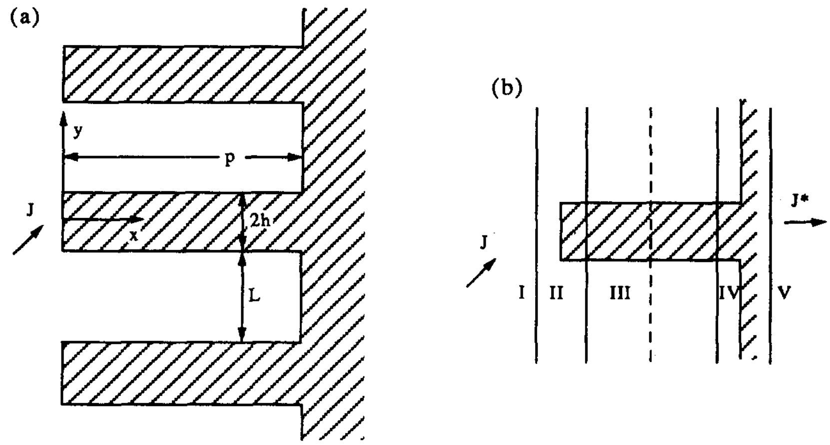

Fig. 4. Saw-type wall with rectangular fins (a) and schematic decomposition into five zones with different temperature patterns (b).

$$
\begin{aligned}
u_1 &= \frac{1}{A}\cos(\lambda\theta_1 - 3\pi\gamma)(c_1 F + c_2/F) \\
v_1 &= \frac{1}{A}\sin(\lambda\theta_1 - 3\pi\gamma)(c_1 F - c_2/F) \\
u_2 &= \cos(\lambda\theta_1 + \pi\gamma)(c_1 F + c_2/F) \\
v_2 &= -\sin(\lambda\theta_1 + \pi\gamma)(c_1 F - c_2/F)
\end{aligned}
\tag{5}
$$

and at $e < 1$

$$
\begin{aligned}
u_1 &= \frac{1}{A}\sin(\lambda\theta_1 - 3\pi\gamma)(c_1 F - c_2/F) \\
v_1 &= \frac{1}{A}\cos(\lambda\theta_1 - 3\pi\gamma)(c_1 F + c_2/F) \\
u_2 &= -\sin(\lambda\theta_1 + \pi\gamma)(c_1 F - c_2/F) \\
v_2 &= -\cos(\lambda\theta_1 + \pi\gamma)(c_1 F + c_2/F).
\end{aligned}
\tag{6}
$$

In equations (5) and (6), subscript 1 corresponds to fins and 2 corresponds to the environment. Besides, we used the following designations:

$$
x_1 = \frac{\pi x}{2h}, \quad y_1 = \frac{\pi y}{2h}, \quad D = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|, A = \frac{k_1 + k_2}{2k_1}
$$

$$
\cos(\pi\gamma) = \frac{\sqrt{2-D}}{2},
$$

$$
\cos(\pi\lambda) = 1 - \frac{D^2}{2}, \quad \theta_0 = \begin{cases} \pi/2, & x > 0 \\ -\pi/2, & x < 0 \end{cases}
$$

$$
\theta_1 = \tan^{-1}\frac{\cos y_1}{\sinh x_1} + \theta_0, \quad F = \left(\frac{\cosh x_1 + \sin y_1}{\cosh x_1 - \sin y_1}\right)^{\lambda/2}
$$

$$
c_1 = \frac{u_\infty}{\sqrt{2+D}} + \frac{v_\infty}{\sqrt{2-D}}, \quad c_2 = \frac{u_\infty}{\sqrt{2+D}} - \frac{v_\infty}{\sqrt{2-D}}. \tag{7}
$$

Thus, equations (5)–(7) present formulae for flux components in terms of physical and geometrical parameters expressed in elementary functions (for more general case of finite p they involve elliptic functions).

From these formulae it is easy to show that at $x \to +\infty$ the flux is

$$
u_1^\infty = \frac{2}{1+e}u_\infty, \quad v_1^\infty = \frac{2}{1+e}v_\infty
$$

wherefrom we conclude that the flux vector at the 'right' infinity in the fins is colinear with one at the 'left' infinity in the environment and their values differ at most twice.

Analogously,

$$
u_2^\infty = \frac{k_2}{k_1}u_1^\infty, \quad v_2^\infty = v_1^\infty
$$

which represents the trivial pattern of 1-D flux refraction in a layered medium sufficiently far right from the fin tips.

To plot the streamlines we integrate the system of ordinary differential equations

$$
\mathrm{d}x/\mathrm{d}t = u(x,y), \quad \mathrm{d}y/\mathrm{d}t = v(x,y) \tag{8}
$$

using the routine fourth-order Runge–Kutta method.

In Fig. 5 streamlines are shown for one strip of our cascade at $u_1^\infty = v_1^\infty = 0.5$ and (a) $k_1 = 10$, $k_2 = 1$ (b) $k_1 = 1$, $k_2 = 5$, respectively. To plot these lines we selected 10 points at $y/h = -2$, $x/h = -2+0.02\ m$, $m = \overline{1,10}$ and traced them till $y/h = 2$ according to equation (9).

Figure 6 illustrates in a larger scale one element of the fin with nontrivial streamline pattern. Namely, we traced again particles with initial conditions $y/h = -1.5$, $x/h = -0.2+0.05\ m$ and split out the region $-0.2 < x < 0.4$. Trajectories 1–4 show 'reversal' of streamlines near the tip edge. Clearly, initial conditions in equation (8) can be chosen arbitrarily.

If we return to the general case of finite p from intuition and calculations made we infer that at sufficiently high p/h value temperature in our 'saw-

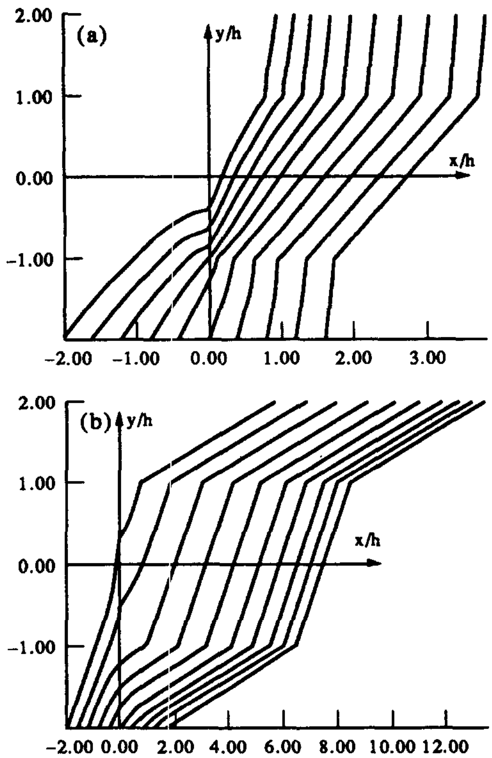

Fig. 5. Streamlines near a strip at $u_1^\infty = v_1^\infty = 0.5$ and: (a) $k_1 = 10, k_2 = 1$; (b) $k_1 = 1, k_2 = 5$.

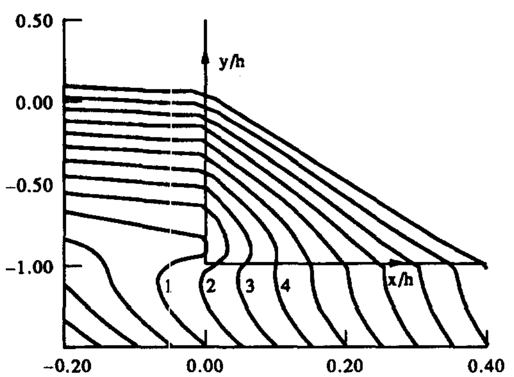

Fig. 6. Finer streamline picture near the edge point of a strip.

type' structure can be decomposed into five zones (Fig. 4b). Zones I and V represent 1-D field in the homogeneous wall and environment. In zone III we have routine refraction in a layered medium. Zones II and IV are mirror-symmetric and exhibit essentially 2-D fields near the fin tips and roots, respectively. Obviously, in the large scale, i.e. sufficiently far from the grooves, flux components in zones I and V are related through the standard refraction condition along an effectively flat boundary, shown in Fig. 4b by a dashed line.

What is the advantage of solutions above as compared with standard FDM or FEM procedures? Numerical solutions to equation (1) are usually performed for discrete meshes with *a posteriori* approximation of fluxes while equations (5)–(7) are directly expressed in terms of gradient. Obviously, any real cooling structure exhibits more complex geometry than we studied and numerical approaches should be used. Analytical solutions serve as check points for computer codes and as a means to elucidate nontrivial peculiarities of temperature fields which can be otherwise blurred or overlooked.

5. CONCLUSIONS

Problems of steady 2-D heat conduction from developed surfaces are solved in explicit rigorous form. For high conductivity of the wall material optimal shape design, for equidistantly spaced fins, is performed. The fin confines extreme area at prescribed total fluxes from the fin and the whole period of the system is shown to coincide with the known contour of a concrete dam of constant hydraulic gradient determined by Polubarinova-Kochina [16]. This unique global extremum is found in the class of arbitrary curves. In the case of finite ratio of conductivities of the wall and environment optimization is made for a single fin and in the class of elliptic curves. Depending on the ratio two nontrivial local extrema exist, a maximum and a minimum. For a system of semi-infinite rectangular fins with arbitrary ratio and orientation of the outer gradient at infinity, thermal gradient distribution is determined. This problem involves two harmonic fields which conjugate by two conditions of temperature and normal flux continuity along the periodic interfaces.

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